# Approach to Modeling and Estimation for Uncertain Systems

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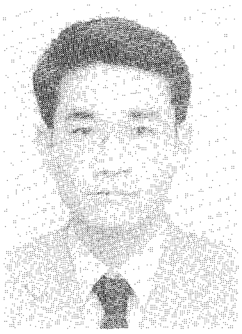
**Optimal modeling and optimal estimation are discussed for uncertain dynamic systems. Uncertainty of the system is assumed to be presented by multiple models, or a set of possible parameters, and their own probabilities. Optimal solutions are derived by minimizing cost functions of quadratic form. If the number of the multiple models is finite, the optimal nominal model and the optimal estimator can be realized by systems of finite order. In general, however, the order of these optimal solutions is much higher than that of each possible dynamic model. In order to make their implementation more practical, approximation by reduced-order models is necessary. These problems can be reduced to those of ordinary reduced-order modeling and reduced-order estimation, where some computation is necessary. Suboptimal reduced-order models and estimators that can be obtained with simple calculation are also investigated. Approaches developed recently, including the balanced realization model, the chained aggregation model, and the principal coordinate realization model are applied. These approaches are compared by using simple examples. In the examples, performance robustness obtained by the multiple model method is also illustrated.**

## I. Introduction

IN the analysis and synthesis of a dynamic system, uncertainties in the given dynamical model or parameter changes due to various environmental conditions are often present. The multiple model approach is one of the direct methods to treat such systems, but it is mainly applied to the feedback control problem.[1-3] Its application to the modeling and estimation problem has not been discussed. The analytical results are simpler than those of the control problem, because of its open-loop nature. In the modeling and estimation problem, there are also cases in which performance robustness is considerably enhanced by the multimodel approach.

In the first part of the paper, modeling of uncertain system is discussed. The nominal model that represents a set of multimodels is derived from the minimization of the squared output errors between the actual system and the model. When the actual system is assumed to be a realization of one of the possible dynamic models, the response of the best nominal model becomes a weighted average of that of each possible model. If the number of possible models is large, this full-order optimal nominal model becomes a very high-order system. In practical application, this complex full-order model is not necessary, and a reduced-order model or a nominal model of fixed order will be useful. The problem of order



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reduction for a multimodel system is equivalent to that for an ordinary system, and results for the optimal reduced-order model[4,5] can be applied to this problem. Suboptimal reduced-order models are often useful because of their good approximation results and smaller computational demands.[6-9]

In the latter part of this paper, the design of an estimator for uncertain systems is discussed. In the design of estimators, or filters, designers sometimes face a system in which the performance of the estimator is sensitive to the parameter changes of the system. Therefore, when the estimator is introduced from minimizing estimation errors based on a given nominal model, the performance with off-nominal parameters should be another concern in a practical design. Uncertainty of the dynamic system or parameter changes of the system are unavoidable problems, and performance robustness is an important factor in designing estimators.

This paper proposes an approach for the analysis and synthesis of a robust estimator, the performance of which is insensitive to a parameter change. The performance robustness is directly considered by using multiple models of the estimated system. The optimal estimator is introduced by considering an equivalent system that has the same cost function with a multimodel system. Since the equivalent system is constructed with multiple models, the optimal estimator for the multimodel system can be realized by a system of finite order. However, as in the modeling problem, the optimal full-order estimator generally becomes a high-order system in proportion to the number of given models. The complex full-order estimator is not necessary and a reduced-order estimator is useful in a practical implementation. The problem of the reduced-order estimator for multimodels is that of the ordinary reduced-order estimator, and the optimal solution needs some computational cost.[10,11] Suboptimal reduced-order estimators that can be obtained with simple calculation are considered. Since the problem of the reduced-order estimator can be reduced to that of the reduced-order model, the approaches proposed in the first part of the paper can be applied to this problem, that is, the balanced realization[6] and the chained aggregation with minimum truncation error.[8,9] Another approach that has been proposed recently for the reduced-order estimator problem, the principal coordinate realization,[12] is also applied.

Some simple examples are shown to illustrate the effectiveness of the proposed approach for the modeling and estimation of uncertain systems. These examples indicate that this approach has potential applicability to aerospace engineering, e.g., on-line estimation for flexible space structures, aeroelastic systems, and so on.

## II. Optimal Nominal Model for a System with Uncertain Parameters

### Problem Statement and the Optimal Solution

The system to be modeled is considered to have uncertainty with respect to its realization among $M$ possible dynamics. A probability for the realization of each possible dynamics is assumed to be given as $p_i$ for the $i$th model, where

$$\sum_{i=1}^{M} p_i = 1$$

The dynamics of each model are given by the following state equation:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = A_i x_i(t) + B_i u(t)$$
$$y_i(t) = C_i x_i(t) \tag{1}$$

where $u \in R^m$, $y_i \in R^q$, $x_i \in R^{n_i}$ are the input, output, and state of each system, respectively. The system matrices $(A_i, B_i, C_i)$ are constant matrices of adequate dimension. It is assumed that each system is asymptotically stable, $(A_i, B_i)$ is controllable, and $(C_i, A_i)$ is observable. The transfer functions of the nominal model and the $i$th model are defined by $\Phi(s) \in C^{q \times m}$ and $\Phi_i(s) \in C^{q \times m}$, respectively. From Eq. (1), the latter is given by

$$\Phi_i(s) = C_i(sI_{n_i} - A_i)^{-1} B_i \tag{2}$$

The nominal model that represents the set of all possible dynamics is derived by minimizing a cost function. For the $i$th realization, the cost function is defined by the following quadratic form:

$$J_i = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \mathrm{tr}\{Q[\Phi_i(s) - \Phi(s)] \times W(s)[\Phi_i(-s) - \Phi(-s)]^T\} \mathrm{d}s \tag{3}$$

where a positive-definite constant matrix $Q \in R^{q \times q}$ is a weight on the output error and a Hermitian positive-definite matrix $W(j\omega) \in C^{m \times m}$ is a weight on the input.

The cost function defined with frequency responses (3) can be interpreted as having another physical meaning by using Parseval's theorem. When the input $u(t)$ is assumed to be random noise, the power spectrum of which is $W(j\omega)$, the cost function (3) is equal to the following:

$$J_i = E\{[y_i(t) - y(t)]^T Q [y_i(t) - y(t)]\} \tag{4}$$

where $E[\;]$ denotes the time average and $y(t)$ is the output of the nominal model. In this paper, all random input is assumed to be stationary, Gaussian, and of zero mean.

If we consider the probability of the $i$th realization, the expectation of the cost function or the total cost function can be written as follows:

$$J = \sum_{i=1}^{M} p_i J_i \tag{5}$$

The optimal solution of $\Phi(s)$ that minimizes the cost function in Eq. (5) can be derived from the variation of the transfer function. The optimal solution is given by

$$\Phi_0(s) = \sum_{i=1}^{M} p_i \Phi_i(s) \tag{6}$$

The optimal transfer function is merely the weighted summation of each transfer function, and it does not depend on the power spectrum of the input nor the weight on the output. This optimal solution can be easily realized by a state equation that is constructed from the state equations of each dynamic in Eq. (1) as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + Bu(t), \quad x \in R^N, \quad N = \sum_{i=1}^{M} n_i$$
$$y_0(t) = Cx(t) \tag{7}$$

where the system matrices $(A, B, C)$ are

$$A = \text{block diag}(A_1, A_2, \ldots, A_M) \in R^{N \times N}$$
$$B^T = [B_1^T, B_2^T, \ldots, B_M^T] \in R^{m \times N}$$
$$C = [p_1 C_1, p_2 C_2, \ldots, p_M C_M] \in R^{q \times N}$$

where $y_0 \in R^q$ is the output of the optimal nominal model.

The optimal nominal model is realized by a system of order $N$.

$$N\left(\sum_{i=1}^{M} n_i\right)$$

In general, the order of the optimal model is much higher than each possible model, and thus a realization of minimal order and the approximation by a reduced-order model of reasonable order are worthy of investigation.

**Reduced-Order Nominal Model**

Although each model is assumed to be controllable and observable, the optimal solution (7) is not necessarily of minimal realization. If the system matrices of each model $A_i$ have some common eigenvalues, the optimal model (6) can be realized by a system of order less than $N$. Minimal realization can be also obtained in the process of the order reduction of the model, which is discussed in the following.

When there is a constraint on the order of the nominal model, the cost function is better written as follows:

$$J = J_0 + J_r$$

$$J_r = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}\{Q[\Phi(s)-\Phi_0(s)]W(s) \times [\Phi(-s)-\Phi_0(-s)]^T\}\,\mathrm{d}s \tag{8}$$

where $J_0$ is the cost function that is obtained by the full-order optimal model $\Phi_0$, and $\Phi(s)$ is the transfer function of the reduced-order nominal model. The cost function (8) shows that the nominal model with a constraint on its order can be derived from the full-order optimal model $\Phi_0(s)$. The problem is reduced to that of an ordinary reduced-order model problem.

The reduced-order model of the least-square output error has been discussed in various references.[4,5] Hyland proposed an optimality condition and a computational algorithm for the case in which the input is white noise.[5] It consists of an iterative calculation and convergence to the global minimum depends on the initial solution. When the input contains colored noise, or $W$ is not a constant matrix, such an optimality condition or computational algorithm has not been developed. In order to obtain the optimal solution, some sort of gradient method of parameter optimization would be necessary. With any approach adopted, it needs an adequate initial solution or a suboptimal reduced-order model. When the suboptimal reduced-order model gives a performance close to the full-order model, the optimization process can be omitted. Therefore, the suboptimal reduced-order model that can be obtained with simple calculation is important. In this paper, the balanced realization model and the chained aggregation model are applied to this problem. The computational demands for both models are quite small, and they give unique models without any free parameters. These methods are based on the internal presentation of the state equation, where the significance of each state on the input-output relation is explicitly expressed.

In these methods, the covariance matrix of the state of the full-order model (7), $X$, is necessary. In order to make the calculation with matrix operations, the input power spectrum

$$\frac{\mathrm{d}x_w}{\mathrm{d}t} = A_w x_w(t) + B_w w_1(t)$$

$$u(t) = C_w x_w(t) + w_2(t) \tag{9}$$

where $w_1$, $w_2$ are white noise, with intensities $W_1$ and $W_2$, i.e.,

$$E\left\{\begin{bmatrix} w_1(t) \\ w_2(t)\end{bmatrix}[w_1^T(t+\tau),\ w_2^T(t+\tau)]\right\} = \begin{bmatrix} W_1 & 0 \\ 0 & W_2\end{bmatrix}\delta(\tau) \tag{10}$$

The shaping filter is assumed to be asymptotically stable and of minimal order. The power spectrum of the shaping filter output is

$$W(s) = C_w(sI-A_w)^{-1}B_w W_1 B_w^T(-sI-A_w)^{-T}C_w^T + W_2,\quad s=j\omega \tag{11}$$

The covariance matrix for the state can be obtained from the following partitioned Lyapunov matrix equations:

$$E\left\{\begin{bmatrix} x(t) \\ x_w(t)\end{bmatrix}[x^T(t),\ x_w^T(t)]\right\} = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3\end{bmatrix}$$

$$AX_1 + X_1A^T + BC_wX_2^T + X_2C_w^TB^T + BW_2B^T = 0$$

$$AX_2 + X_2A_w^T + BC_wX_3 = 0$$

$$A_wX_3 + X_3A_w^T + B_wW_1B_w^T = 0 \tag{12}$$

**Balanced Realization Model**

The balanced realization model is based on the two gramians, the controllability gramian and the observability gramian. The observability gramian $Y_1$ is given by the following equation:

$$A^TY_1 + Y_1A + C^TQC = 0,\quad Y_1 \in R^{N\times N} \tag{13}$$

where $Q$ is the weight on the output in Eq. (4). The controllability gramian is equal to the covariance matrix, when the input $u$ is white noise Here, the balanced realization is extended to the case of a colored noise input by using the covariance matrix $X_1$ for a controllability gramian. The balanced realization that is obtained by the state transformation $T$ satisfies the following condition:

$$TX_1T^T = T^{-T}Y_1T^{-1} = \Sigma$$

$$\Sigma = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2)$$

$$\sigma_1^2 \ge \sigma_2^2 \ge \ldots \ge \sigma_N^2 > 0 \tag{14}$$

where $T$ is a nonsingular matrix.

The reduced-order model is derived by truncating less important states in the transformed state $Tx$. Its state-space equation and system matrices $(A_r, B_r, C_r)$ are given as follows:

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}t} = A_r\xi_r(t) + B_ru(t),\quad \xi_r \in R^r$$

$$y_R(t) = C_r\xi_r(t)$$

$$TAT^{-1} = \begin{bmatrix} A_r & X \\ X & X\end{bmatrix},\quad TB = \begin{bmatrix} B_r \\ X\end{bmatrix},\quad CT^{-1} = [C_r\ X] \tag{15}$$

**Chained Aggregation Model with Minimum Truncation Error**

The reduced-order model is introduced from the realization that is obtained by the state transformation $Tx$. The realization satisfies the following conditions:

$$TX_1T^T = I_N,\qquad CT^{-1} = [C_a, 0, \ldots, 0]$$

$$TAT^{-1};\quad \text{a generalized Hessenberg matrix} \tag{16}$$

The reduced-order models $(A_r, B_r, C_r)$ are derived in the same way as in Eq. (15). With the input as white noise, the reduced-order model was proposed and discussed in Ref. 8, where it was introduced from conditions of matching output covariances and Markov parameters. The realization also can be obtained from the criterion of minimum equation errors in the process of constructing the state equations. (Details are discussed in Ref. 9). Even when the input is not white noise, the realization can be extended by using the covariance matrix $X_1$. In this case, the condition of matching output covariances is not maintained, but the minimality condition of equation errors is still satisfied.

The conditions of Eqs. (14) and (16) are based on the assumption that $X_1$ and $Y_1$ are positive-definite. When the realization of Eq. (7) is not of minimal order, $X_1$ and/or $Y_1$ become positive-semidefinite. The eigenspaces that correspond to the zero eigenvalues are unnecessary, and they can be deleted. In the numerical computation, however, the same algorithm based on Eq. (14) or (16) can be used for such cases.

## III. Optimal Estimator for a System with Uncertain Parameters

### Problem Statement and the Optimal Solution

The dynamic system, or plant, to be estimated is described by the following state equation:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = A_i x_i(t) + B_i w(t)$$
$$y_i(t) = C_i x_i(t) + v(t)$$
$$\eta_i(t) = D_i x_i(t) \tag{17}$$

The plant is excited by the disturbance $w \in R^k$, which is assumed to be white noise of intensity $W$. The measurement is $y_i \in R^m$, which is contaminated by white noise $v$, with intensity $V$. These random processes are assumed to be stationary Gaussian noise of zero mean. The constant matrices $A_i$, $B_i$, $C_i$, and $D_i$, have adequate dimensions. It is assumed that $(A_i, B_i)$ is controllable $(C_i, A_i)$ is observable, and the system is asymptotically stable. The plant dynamics have uncertainty, and they are assumed to be represented by $M$ multiple models. The subscript $i$ denotes the $i$th model of the plant dynamics $(i=1,M)$. The probability of realization for each dynamic is given by $p_i$, where

$$\sum_{i=1}^{M} p_i = 1$$

$\eta_i \in R^q$ is the variable to be estimated from the measurement. The estimator problem considered in this paper is illustrated in the block diagram of Fig. 1, where variables are presented in the Laplace domain. The transfer functions $F_i$ and $G_i$ are given by

$$F_i(s) = D_i(sI - A_i)^{-1}B_i \quad \in C^{q\times k}$$
$$G_i(s) = C_i(sI - A_i)^{-1}B_i \quad \in C^{m\times k} \tag{18}$$

$K(s)$ is the estimator transfer function, and it should generate the best estimate $\hat{\eta}_i$ for $\eta_i$ from the measurement $y_i$.

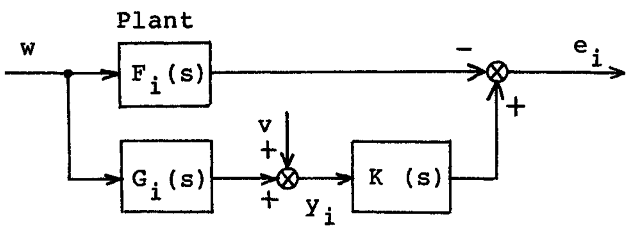


**Fig. 1 Block diagram of estimation for multimodel systems.**

The cost function for each dynamic is defined by the quadratic function of estimation errors, i.e.,

$$J_i = E[e_i^T Q e_i], \qquad e_i(t) = \hat{\eta}_i(t) - \eta_i(t) \tag{19}$$

where the weighting matrix $Q$ is an appropriately given positive-definite matrix.

The cost function $J_i$ can be written with the transfer functions

$$J_i = \left(\frac{1}{2\pi j}\right)\int_{-j\infty}^{j\infty} \mathrm{tr}(Q\{[F_i(s) - K(s)G_i(s)]W \times [F_i(-s) - K(-s)G_i(-s)]^T + K(s)VK(-s)^T\})\,\mathrm{d}s \tag{20}$$

If we consider the probability of each dynamic, the expectation of the cost function or the total cost can be defined by the following:

$$J = \sum_{i=1}^{M} p_i J_i \tag{21}$$

The optimal estimator that minimizes the cost function of Eq. (21) can be derived by considering an equivalent system that generates the same cost function as Eq. (21). The equivalent system is illustrated in Fig. 2. In this equivalent system, the dynamics for the measurement $y$ and the variable to be estimated $\eta$ are given by

$$y(s) = G(s)w_0(s), \qquad G(s) = [G_1(s),\ldots,G_M(s)]$$
$$\eta(s) = F(s)w_0(s), \qquad F(s) = [F_1(s),\ldots,F_M(s)] \tag{22}$$

where $w_0^T(s) = [w_1^T(s),\ldots,w_M^T(s)]$. The estimation error is given by

$$e(s) = [F(s) - K(s)G(s)]w_0(s) + K(s)v(s) \tag{23}$$

The newly defined disturbances $w_i(t) \in R^k$ $(i=1,M)$ are white noise and they are not correlated with each other. The intensity of each disturbance $W_i$ is given by

$$W_i = p_i W \tag{24}$$

Then, the cost function of this combined system is written as

$$J = \left(\frac{1}{2\pi j}\right)\int_{-j\infty}^{j\infty} \mathrm{tr}(Q\{[F(s) - K(s)G(s)]W_0 \times [F(-s) - K(-s)G(-s)]^T + K(s)VK(-s)^T\})\,\mathrm{d}s \tag{25}$$

where $W_0 = \text{block diag}(p_1 W,\ldots,p_M W)$.

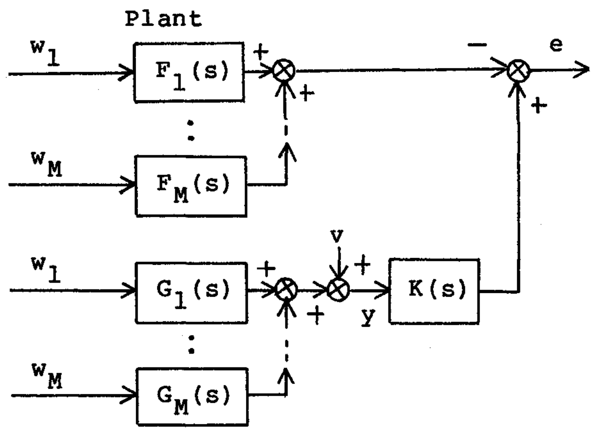


**Fig. 2 Block diagram of an equivalent system.**

Since the cost function (25) is equal to Eqs. (20) and (21), the optimal estimator for this equivalent system is also optimal for the multidynamic system in Eq. (18). Therefore, the optimal estimator for the multidynamic system is realized by a system of finite order.

Since the optimal estimator can be calculated easier with a state-space equation than with a transfer function, the following discussion will be carried out in the time domain. The equivalent system can be simply expressed by the following state equation:

$$\frac{dx}{dt} = Ax(t) + Bw_0(t)$$

$$y(t) = Cx(t) + v(t) \tag{26}$$

where

$$A = \text{block diag}(A_1, A_2, \ldots, A_M)$$

$$B = \begin{bmatrix} B_1, 0, \ldots, 0 \\ 0, B_2, 0, \ldots, 0 \\ 0, \ldots, B_M \end{bmatrix}$$

$$C = [C_1, \ldots, C_M]$$

where $x \epsilon R^{N \times N}$ is the state and $N$ is equal to

$$\sum_{i=1}^{M} n_i$$

The optimal estimator for system (26) is given by the following:

$$\frac{d\hat{x}}{dt} = A\hat{x}(t) + K[y(t) - C\hat{x}(t)]$$

$$\hat{\eta}(t) = D\hat{x}(t)$$

$$D = [D_1, \ldots, D_M] \tag{27}$$

The Kalman gain $K$ is given by the following equation:

$$AX + XA^T + BW_0B^T - XC^TV^{-1}CX = 0, \quad K = XC^TV^{-1} \tag{28}$$

The solution of the steady-state matrix Riccati equation $X$ is the covariance matrix of the estimation error, i.e.,

$$X = E[ee^T], \qquad e(t) = \hat{x}(t) - x(t) \tag{29}$$

The covariance matrices of the state and its estimate are defined

$$E[xx^T] = X_1, \qquad E[\hat{x}\hat{x}^T] = X_2 \tag{30}$$

These matrices are given by the following matrix equations:

$$AX_1 + X_1A^T + BW_0B^T = 0, \qquad X_2 = X_1 - X \tag{31}$$

Equation (31) comes from the characteristic of the optimal estimation that the estimate and the error are uncorrelated.

As in the nominal model problem, the optimal estimator is of a higher order compared with each given model, and the minimal realization and reduced-order (or simplified) estimator is worthy of investigation for its practical implementation.

### Minimal Realization

Since each model is assumed to be controllable, the equivalent system Eq. (26) is also controllable. However, the observability of the equivalent system is not necessarily maintained. Even if there are some unobservable spaces in the equivalent system, since each model is assumed to be asymptotically stable, the equivalent system is detectable. Then the matrix Riccati equation has a unique positive-definite solution, and the full-order optimal estimator (27) can be uniquely determined. However, this does not guarantee its minimal realization. When the covariance matrix of the state estimate $X_2$ is positive-semidefinite, it means the estimator [Eq. (27)] contains undisturbed space, so it can be realized with a system of reduced order that is equal to rank $(X_2)$. Since the innovation process $y$-$C\hat{x}$ becomes white noise of intensity $V$ in the optimal estimation, $X_2$ is equal to a controllability gramian for the system characterized by the matrices $(A, K)$, and the undisturbed space in $X_2$ corresponds to the uncontrollable space of the optimal estimator.

For general cases, the minimal-order estimator can be obtained in the process of constructing the suboptimal reduced-order estimator; this is discussed later. To illustrate the minimal realization, one extreme case in which the order of the optimal estimator does not depend on the number of possible models is discussed here. The uncertain model is defined such that the system matrices $A_i$ are common for all possible models, i.e.,

$$A_i = A_1, \quad D_i = I_n, \quad n_i = n; \quad i = 1, M \tag{32}$$

The input matrix $B_i$ and/or the output matrix $C_i$ have uncertainty. Although the simply combined system (26) and its full-order estimator (27) are of order $N(=n \times M)$, the optimal estimator can be realized by a system of order $n \times \min(M, m)$. When the number of measurement is less than the number of a possible model, i.e., $m < M$, the same performance with the simply obtained full-order estimator of Eq. (27) can be realized by a reduced-order estimator. This can be explained by using the estimator dynamics of Eq. (27). The controllability of the estimator can be determined by the system matrices $(A, K)$. If $m < M$, the dimension of the controllable space is $n \times m$ at most, and the optimal estimator can be realized by this controllable space.

### Reduced-Order Estimator

The term "reduced-order estimator" is used for an estimator with a constraint of its order, and it is different from that defined for the noise-free filter. The optimal reduced-order estimator has also been discussed by Bernstein and Hyland.[10] They have introduced an optimality condition and a computational algorithm. However, as in the modeling problem, the algorithm is based on an iterative calculation without any guarantee of convergence to the global minimum solution, and so it needs an adequate initial solution, or a suboptimal reduced-order estimator. If the suboptimal reduced-order estimator gives a performance close to that of the full-order optimal estimator, the optimization process can be omitted. Therefore, the suboptimal reduced-order estimator is important, but is has not been fully discussed in the literature. At present, it seems to be difficult to single out the best method, and it is better to apply some promising methods for each problem and compare their results.

As in the modeling problem, the suboptimal reduced-order estimator can be derived by truncating less important states of the original full-order estimator. When the state transformation $\xi = Tx$, $\hat{\xi} = T\hat{x}$ is considered, the dynamics of the filter do not change and they can be written by the transformed matrices, where $T$ is an arbitrary nonsingular matrix.

$$\frac{d\hat{\xi}_r}{dt} = TAT^{-1}\hat{\xi}(t) + TK[y(t) - CT^{-1}\hat{\xi}(t)]$$

$$\hat{\eta}(t) = DT^{-1}\hat{\xi}(t) \tag{33}$$

The reduced-order estimator is derived by truncating less important states in the transformed state $T\hat{x}$. The reduced order estimator is given by

$$\frac{d\hat{\xi}_r}{dt}=A_r\hat{\xi}_r(t)+K_r[y(t)-C_r\hat{\xi}_r(t)]$$

$$\hat{\eta}(t)=D_r\hat{\xi}_r(t), \qquad \hat{\xi}_r\epsilon R^r$$

$$TAT^{-1}=\begin{bmatrix} A_r & X \\ X & X \end{bmatrix}, \qquad TK=\begin{bmatrix} K_r \\ X \end{bmatrix}$$

$$CT^{-1}=[C_r,X], \qquad DT^{-1}=[D_r,X] \tag{34}$$

A different internal realization gives a different reduced-order estimator. Three models are applied to this problem.

*Balanced Realization Model*

The observability gramian of the estimator [Eq. (27)] is defined as follows:

$$(A-KC)^TY_2+Y_2(A-KC)+D^TQD=0 \tag{35}$$

Balancing of the covariance matrix of the estimator $X_2$ and the observability gramian $Y_2$ gives the following condition:

$$TX_2T^T=T^{-T}Y_2T^{-1}=\Sigma$$

$$\Sigma=\text{diag}(\sigma_1^2,...,\sigma_N^2), \qquad \sigma_1^2\geq...\geq\sigma_N^2>0 \tag{36}$$

This condition determines the transformation matrix $T$ and introduces a unique realization.

*Chained Aggregation Model*

The chained aggregation model is obtained from a realization introduced by the following conditions:

$$TX_2T^T=I_N, \qquad DT^{-1}=[D_a,0,...,0], \qquad D_a\epsilon R^{q\times q} \tag{37}$$

and the system matrix $T(A-KC)T^{-1}$ is a generalized Hessenberg matrix.

The conditions in Eqs. (36) and (37) are based on the assumption that $X_2$ and $Y_2$ are positive-definite. When the estimator (33) is not of minimal order, $X_2$ and/or $Y_2$ become positive-semidefinite. The eigenspaces that correspond to the zero eigenvalues are unnecessary, and they can be deleted. In the numerical computation, however, the same algorithm based on Eq. (36) or (37) can be used for such cases.

The reduced-order estimator problem can be reduced to the reduced-order modeling problem, and the two models just proposed are equal to the models derived in the first part of the paper. A discussion of this follows.

For computation of the optimal solution, the state-space approach gives an efficient algorithm. However, when the characteristics of the order reduction are discussed, the transfer-function approach gives another perspective to the problem. By using the cost function defined in Eq. (25), the full-order optimal solution $K_0(s)$ satisfies the following equation:

$$[(K_0(s)\{G(s)W_0G^T(-s)+V\}-F(s)W_0G^T(-s)]_+=0 \tag{38}$$

Table 1 Parameters for multiple models of example 1

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\omega_i$ | 0.85 | 0.9 | 0.95 | 1.0 | 1.05 | 1.1 | 1.15 |
| $p_i$ | $1/2^6$ | $6/2^6$ | $15/2^6$ | $20/2^6$ | $15/2^6$ | $6/2^6$ | $1/2^6$ |

$[\ ]_+$ denotes the analytic part on the right-hand side of the $s$ plane. When there is a constraint on the order of the estimator $K(s)$, the cost function in Eq. (25) is better rewritten by using the optimality condition.

$$J=J_0+J_r$$

$$J_r=\left(\frac{1}{2\pi j}\right)\int_{-j\infty}^{j\infty}\text{tr}\{Q[K(s)-K_0(s)]$$

$$\times[G(s)W_0G^T(-s)+V][K(-s)-K_0(-s)]^T\}ds \tag{39}$$

$J_0$ is the cost function obtained by the full-order optimal solution $K_0(s)$. $J_r$ is the increment due to the nonoptimal transfer function of $K(s)$. This cost function shows that the estimator with a constraint on its order is the optimal reduced-order model for the full-order optimal estimator $K_0(s)$, the input of which is colored noise and its power spectrum is equal to that of the measurement. Therefore, the models derived for the reduced-order nominal model can be applied to the reduced-order estimator problem.

*Principal Coordinate Realization Model*

Since the optimal estimator has the characteristic that the estimated state and its error are uncorrelated, there is a state transformation that gives diagonalized covariance matrices of the state, its estimate, and its error, i.e.,

$$TX_1T^T=I_N$$

$$TX_2T^T=\text{diag}(\mu_1^2,...,\mu_N^2)$$

$$TXT^T=\text{diag}(1-\mu_1^2,...,1-\mu_N^2)$$

$$1>\mu_1^2\geq\mu_2^2\geq...\geq\mu_N^2\geq 0 \tag{40}$$

The singular values of this realization $\mu_i^2$ explicitly show the accuracy of the estimation of each state, and they also indicate coupling intensities between each pair of states. Therefore, the realization can be used for the order reduction of the estimator. Details of this approach are discussed in Ref. 12.

## IV. Examples

**Modeling of an Oscillatory Dynamic System with Natural Frequency Uncertainty**

The optimal modeling problem is applied to a weakly damped oscillatory system. The dynamics are given by the following transfer function:

$$y/u=s/(s^2+0.1s+\omega_i^2) \tag{41}$$

The natural frequency $\omega_i$ is uncertain and it is represented by seven possible models. The values and their probabilities are given in Table 1. Figure 3 shows the impulse responses of the nominal models. Compared with the nominal model of center value ($\omega=1$), the optimal full-order model has dynamics with larger damping. Since the effect of the uncertainty in the natural frequency increases with time, the optimal model damps more quickly.

The minimum absolute value of the cost function is $J_0=1.5328$, ($W=1$, $Q=1$), which is realized by the full-order model of the fourteenth order. When there is a constraint on its order, the suboptimal reduced-order models obtained by the balanced realization and the chained aggregation give the cost functions shown in Fig. 4. The increment due to the order reduction is extremely small, especially for those cases in which the order is more than four. In this case, the balanced realization model gives better performance for lower-order reduction cases. If we compare the cost function obtained by the nominal model of center value ($\omega=1$); $J/J_0=1.377$, the balanced realization reduced-order model of second order can be considered a very good approximation as a nominal model. Its impulse response is also plotted in Fig. 3.

**Integration Filter Designed with the Plant Dynamics**

Two simple examples for the estimation of uncertain systems are considered. One example is introduced in the construction of a velocity component from an acceleration measurement, where the estimation filter is derived by using the plant dynamics and noise intensities.

This example is motivated by a wind-tunnel experiment for a gust-load alleviation system.[13] In the experiment, the gust-load alleviation is mainly achieved by generating a damping force proportional to the velocity of the deformation at the control surface position, where the measurement is by an accelerometer. The dominant component of the feedback filter is time integration of the measured acceleration. Although the filter is mildly stable, it becomes sensitive to model error in the low-frequency range. In the actual experimental model, the accelerometer output contains a gravitational force effect, which is proportional to the wing deformation. The feedback control law designed disregarding this effect results in aeroservoelastic divergence, and a modification to increase the stability of the filter is necessary. The key point of this problem is in the process of constructing the velocity from the accelerometer output, which consists of not only acceleration but an uncertain deflection component. This point can be illustrated by the following simplified example.

The dynamics of the model are described by a second-order system that represents the first bending mode motion of the wing model. The gust disturbance is considered to be generated by a shaping filter of the first order. The transfer function of the wing deformation $x_1$ from the disturbance, which includes the shaping filter, is as follows:

$$x_1/w = a/(s+a)/(s^2+2\zeta_1\omega_1 s+\omega_1^2) \tag{42}$$

where $2\zeta_1\omega_1 = 15$, $\omega_1^2 = 400$, and the cutoff frequency of the shaping filter is 1 Hz, i.e., $a = 6.283$.

The measurement $y_1$ (accelerometer output) and the variable to be estimated $\eta_1$ (velocity of the wing deformation) are written as follows:

$$y_1 = \ddot{x}_1 - \omega_1^2\beta x_1 + v_1$$

$$\eta_1 = \dot{x}_1 \tag{43}$$

where a constant component is assumed to be subtracted from the accelerometer output.

The second term of $y_1$ is due to the effect of gravitational force, where $\beta$ depends on the gradient of the mode shape. Here, it is assumed that $\beta$ has the following possibilities:

$$\beta_i = (i-2)\times 0.05, \qquad p_i = 1/3, \qquad i = 1, 3 \tag{44}$$

Each value of $\beta_i$, ($i = 1,3$) corresponds to a possible position of the wing model, hanging down vertically, resting horizontally, or standing up vertically. The intensities of the noise are defined as $W_1 = 100$, and $V_1 = 0.001$ from a physical consideration.

The results for estimation error $E[(\hat{\eta}_1 - \eta_1)^2]$ due to the change of the parameter $\beta$ are shown in Fig. 5. The performance of the filter designed by assuming $\beta = 0$ is also plotted for comparison. This filter gives extremely good accuracy at the design point $\beta = 0$, but the performance deteriorates rapidly in the off-design region. On the other hand, the filter that is designed to minimize the estimation error at three points ($\beta = -0.05$, 0, 0.05) is more insensitive to parameter change with a small sacrifice of performance around $\beta = 0$.

The full-order optimal estimator is realized by a filter of the third order. Since the dynamics of this example can be presented with a common system matrix $A_i$, as discussed earlier, the optimal estimator can be realized by a system of order $n$

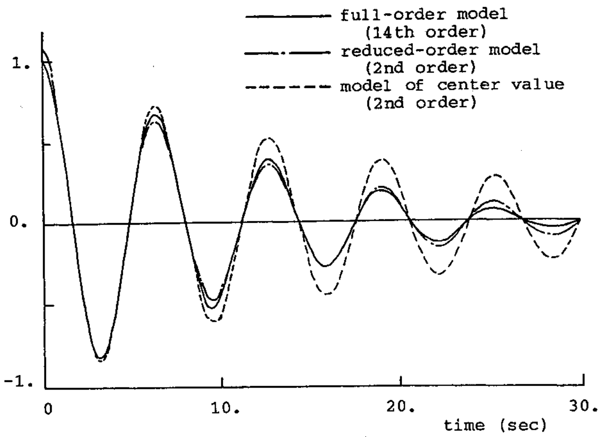


**Fig. 3 Impulse responses of nominal models.**

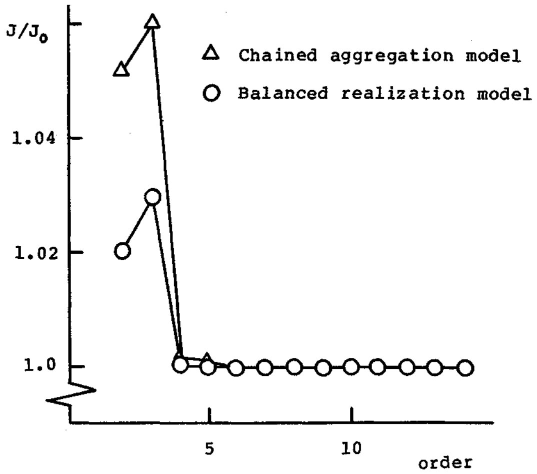


**Fig. 4 Cost functions of reduced-order models.**

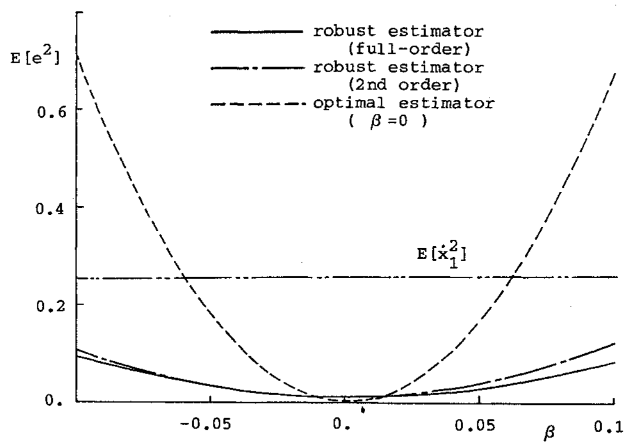


**Fig. 5 Estimation error due to parameter change.**

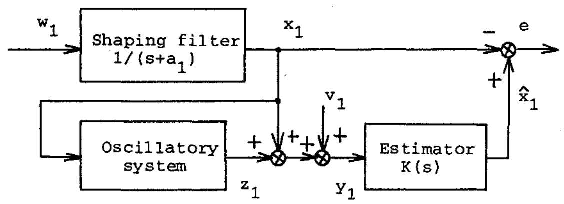


**Fig. 6 Block diagram of notch filter example.**

(the order of each possible model), and it does not depend on the number of possible models. In the figure, the suboptimal filter of the second order that is obtained by the principal coordinate method is also shown. Although the total cost of the estimation error is slightly increased, it gives a performance similar to the full-order (third-order) estimator.

**Notch Filter Design with Natural Frequency Uncertainty**

The system is illustrated by the block diagram in Fig. 6, where all variables are scalars. The signal $x_1$ that is to be estimated is contaminated with noise from a parasitic oscillatory system of low damping ratio. The signal $x_1$ is considered to be generated by a shaping filter of first order with the cutoff frequency $a_1 = 0.1$. $w_1$ is white noise, the intensity of which is given by $W_1 = 1$. The measurement $y_1$ is given as

$$y_1 = x_1 + z_1 + v_1 \tag{45}$$

where $v_1$ is measurement noise, and $z_1$ is the output of the oscillatory system driven by the signal $x_1$, the dynamics of which are given by the following second-order system:

$$z_1/x_1 = a_1/(s^2 + 2\zeta_1\omega_1 s + \omega_1^2) \tag{46}$$

The scalar filter $K(s)$ is being designed to estimate the signal $x_1$ from the measurement $y_1$. Here, the natural frequency of the parasitic system $\omega_1$ has some uncertainty, and it is modeled by the following five possible values, i.e.,

$$\omega_{1_i} = 0.9 + 0.03\, i, \qquad p_i = 0.2, \qquad i = 1, 5 \tag{47}$$

where $p_i$ is the probability of each realization. The center value of $\omega_1$ is 1. The damping term coefficient is as fixed $2\zeta_1\omega_1 = 0.1$. The measurement noise $v_1$ is assumed to be white noise of intensity $V_1 = 0.01$. It is considered in order to apply an ordinary time-invariant optimal estimator. The estimator becomes a strictly proper system, and the cutoff frequency is determined by the intensity $V_1$. Here, $V_1$ is considered to be so small that the cutoff frequency is higher than $a_1$ and $\omega_1$, and thus it does not have a significant effect on the analysis of uncertainty in the natural frequency.

The estimators calculated are the optimal estimator designed by assuming $\omega_1 = 1$, the full-order optimal estimator that minimizes the total cost function, suboptimal reduced-order estimators with three different approaches, and the optimized reduced-order estimator.

Figure 7 shows mean squares of the estimation error $e^2$ for each estimator. The optimal estimator designed by assuming $\omega_1 = 1$, denoted as the estimator of center value, is of the third order. It gives the minimal error only when the actual $\omega_1$ is equal to 1. The estimation error increases considerably when $\omega_1$ departs from the nominal value, e.g., when $\omega_1 = 0.94$, the mean square error is about four times that of $\omega_1 = 1$. As in the former example, the performance curve shows that the gradient with parameter deviation around the nominal point $\omega_1 = 1$ is quite small, but it does not necessarily mean good performance for a wide range of parameters. These examples illustrate that parameter sensitivity is not sufficient in some cases for analysis and synthesis to provide the performance robustness of the estimation, and this is one of the motivations for this paper.

The full-order optimal estimator for the multimodel dynamics is realized by a system of the eleventh order. Although the simply combined system is of the fifteenth order, the shaping filter dynamics are common for all candidate models, and this introduces an uncontrollable space of the fourth order in the estimator. The performance curve has a wavy form, and local minimum points correspond to the five evaluated points. If the evaluated points are increased and closely spaced, the performance curve has a more flat, bucket form. The zeros of the filter are distributed near the imaginary axis at the locations corresponding to the evaluated frequency points. The mean square error at $\omega_1 = 1$ is increased by about 40% compared with the minimal error, but the estimation error does not change significantly in the range of the evaluated points, i.e., $0.94 < \omega_1 < 1.06$. The total cost function is decreased to $J_{\min} = 0.1465$, where the cost function for the estimator of center value is $J = 0.2477$.

If we use the eleventh-order optimal estimator as a reference, suboptimal reduced-order estimators are considered with three approaches. Figure 8 shows the total cost for each order reduction. In this example, the chained aggregation model gives the best results for all reduced orders. When the reduced order is greater than or equal to 5, the chained aggregation suboptimal models give nearly the same cost as the optimal full-order estimator. The principal realization model also gives reasonable results for higher orders, but the balanced realization mode does not. In the balanced realization of

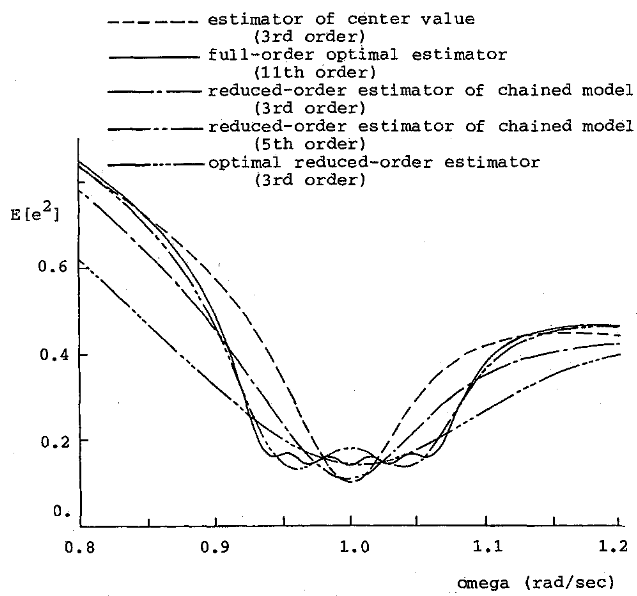


**Fig. 7 Natural frequency effect on estimation errors.**

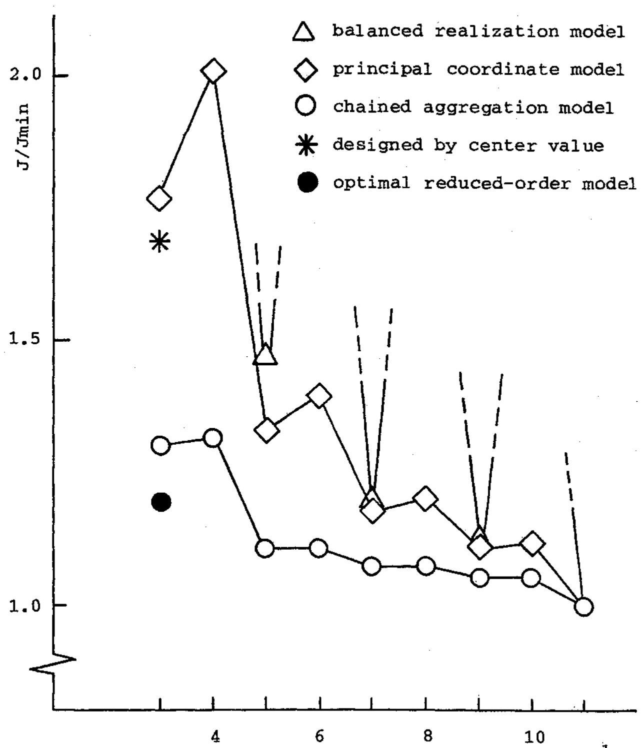


**Fig. 8 Estimation errors of reduced-order models.**

the estimator, $\sigma_i^2$ ($i=1$, 11) is densely distributed in the region $1.26 > \sigma_i^2 > 0.34$, and the significance of each state is not well expressed with the realization. The cost function curves in Fig. 8 show that the suboptimal reduced-order models do not guarantee monotonic characteristics with order reduction. Since the cost function of the optimized estimator decreases monotonically when the order is increased, it can be said that a better suboptimal reduced-order estimator should have such a monotonic tendency.

The optimized estimator of fixed order is also calculated for the third order by using the result of the suboptimal model as an initial solution. The total cost obtained is 18% more than that of the full-order estimator, and it is also plotted in Fig. 8.

In Fig. 7, the performance curves of the estimators, e.g., the third-order optimal estimator, and the third- and fifth-order suboptimal estimators by the chained aggregation approach, are plotted. The optimized third-order estimator gives the flattest curve.

## Conclusions

For an uncertain system, the dynamics of which are represented by a set of possible multiple dynamic systems, the modeling and estimation problems are discussed based on a quadratic cost function. The optimal solution can be realized by a system of finite order and its maximum size is the summation of the orders of each possible dynamic system. In general, however, this full-order optimal solution is a higher-order system than each model, and order reduction is necessary and also possible for practical implementation. The reduced-order model and estimator frequently give a performance that is similar to that of the full-order optimal solution. With the proposed approach, one can efficiently design a robust estimator in which the estimation error is more insensitive to parameter changes in the plant dynamics. Numerical results for simple examples show that performance robustness is gained with this approach and it can be applied to other practical problems.

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